Algebraic Equations with Span less than 4

By Raphael M. Robinson

1. Introduction. It is known [3] that an interval of length greater than 4 must contain infinitely many sets of conjugate algebraic integers, whereas an interval of length less than 4 can contain but a finite number of such sets. The problem remains open for intervals of length 4, except when the end points are integers, in which case there are infinitely many sets of conjugates. For example, the interval $[-2, 2]$ contains the numbers

$$x = 2 \cos 2k\pi/m \qquad [0 \leqq k \leqq m/2, \quad (k, m) = 1],$$

which are a set of conjugate algebraic integers. Kronecker [1] showed that there are no other algebraic integers which lie with their conjugates in $[-2, 2]$.

By the span of a polynomial $f(x)$ or of an algebraic equation $f(x) = 0$ having only real roots, we mean the difference between the largest and smallest roots. We consider irreducible equations with integer coefficients and leading coefficient 1, so that the roots are a set of conjugate algebraic integers. Two such equations are considered equivalent if the roots of one can be obtained from the roots of the other by adding an integer, changing signs, or both. From what we have said, it follows that there are infinitely many inequivalent algebraic equations with span less than 4, but, if $\epsilon > 0$, only a finite number with span less than $4 - \epsilon$.

Thus it appears that algebraic equations with span less than 4 are of particular interest. There are, of course, only a finite number of inequivalent equations of a given degree. We have attempted to make a list of representative equations, one chosen from each class of equivalent equations, with degrees from 2 to 8. The representative may always be chosen in such a way that the average of the roots lies in $[0, \frac{1}{2}]$, and this restriction was accepted. If the average of the roots is 0 or $\frac{1}{2}$, then an ambiguity remains; the equation whose roots are in the shortest possible interval centered at $\frac{1}{4}$ was chosen. In §2, we describe the method of computation which was used, and in §3, the results which were obtained.

2. Method of Computation. The computations described below were carried out during the period June–August 1961 on an IBM 704 at the Computer Center of the University of California, Berkeley. Single-precision floating point arithmetic (which corresponds to using 8 significant decimal digits) was used in the body of the computation. We had occasion to solve a great many algebraic equations, but in every case, these were equations all of whose roots were real. Newton's method was used throughout.

The problem is to find polynomials

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

with integer coefficients, having real roots and span less than 4. It follows that all

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the derivatives $f'(x), f''(x), \cdots$ also have real roots. Also, the span decreases, and hence remains less than 4. Indeed, we can give [4] a better estimate for the span: The span of $f^{(j)}(x)$ for any such polynomial (at least for $n \leqq 25$) is not greater than in the special cases

$$f(x) = (x-2)^m(x+2)^m \quad \text{if } n = 2m,$$

$$f(x) = (x-2)^{m+1}(x+2)^m \quad \text{if } n = 2m+1.$$

Since we agreed that the average of the roots is to lie in $[0, \frac{1}{2}]$, we see that $-n/2 \leqq a_1 \leqq 0$. We try the values $a_1 = 0, -1, -2, \cdots$ in turn.

The general step in the computation consists in deciding what values should be allowed for a_k after $a_1, a_2, \cdots, a_{k-1}$ have already been chosen. We assume that these coefficients were chosen so that the roots of $f^{(n-k+1)}(x)$ are real. Notice that $f^{(n-k)}(x)$ depends only on $a_1, a_2, \cdots, a_k$, and that a change in a_k changes only the constant term of this polynomial. The values of x which produce relative maxima and minima are the roots of $f^{(n-k+1)}(x)$, and are independent of a_k. Thus to make all the roots of $f^{(n-k)}(x)$ real, we need only insist that all of its relative maxima are positive and all of its relative minima are negative.

If $k \geqq 3$, this determines a certain interval in which a_k can lie so that $f^{(n-k)}(x)$ will have real roots. (If $k = 2$, we determine only an upper bound for a_k.) We can then test in each case whether the span of $f^{(n-k)}(x)$ is small enough. Specifically, the procedure chosen is as follows: We start a_k at its largest possible value, and store its smallest value. In case $k < n$, then for each value of a_k in turn, if the span of $f^{(n-k)}(x)$ is not too large, we increase k by one unit and repeat the whole procedure.

If $k = n$, and the span of $f(x)$ is not too large, then we have succeeded in finding a polynomial of degree n with span less than 4. If this is irreducible, then it should be printed out. There was, however, a problem of what to do with the reducible polynomials. If they were all printed, the output would be too bulky. On the other hand, they would serve a useful purpose: We could look at them and see if all the expected ones were there, which would serve as a check on our work. As a compromise, some but not all of the reducible polynomials were printed. For example, for $n = 6$ we printed those reducible sextics which are the product of two irreducible cubics.

If the span of $f^{(n-k)}(x)$ is too large, we can proceed to the next smaller value of a_k without further calculation. After completing the calculation for the minimum possible value of a_k, we backtrack. That is, k is reduced one unit, and we proceed to the next case at that level. If k is even, a shortcut is possible. In this case, $f^{(n-k)}(x)$ is a polynomial of even degree k, so that as we decrease a_k and therefore the constant term of the polynomial, the span increases. Thus as soon as we obtain too large a span, we can immediately backtrack. In case $k = 2$, there is no smallest value of a_k for which the roots of $f^{(n-k)}(x)$ are real, but we can still stop when the span gets too large.

The above procedure was carried out in complete detail for $n \leqq 6$. For degrees 7 and 8, the time for the calculation became prohibitive. It was noticed that for $n \leqq 6$, the spans of $f^{(j)}(x)$ which actually occurred were somewhat smaller than the allowed maxima, and the values of the coefficients a_k which actually led to solutions were in general rather near to the center of the allowed interval. Some reducible polynomials of degrees 7 and 8 were also examined. From all of these, guesses were made as to restrictions, both on the values of the coefficients and on the spans, which were stronger than those we could prove, but which were felt to be unlikely to exclude solutions. The calculations were carried out for $n = 7$ and $n = 8$ using these restrictions.

3. Discussion of the Results. The calculation described in §2 produced a total of 96 inequivalent algebraic equations of degrees from 2 to 8 with span less than 4. These were distributed as follows:

Degree	2,	3,	4,	5,	6,	7,	8.
Equations	4,	5,	14,	15,	17,	15,	26.

However, it should be kept in mind that we are not certain that the list is complete for degrees 7 and 8.

After these 96 equations were found, their roots were recomputed to double precision, and the discriminants of the equations were computed by multiplying the differences of the roots and squaring. In every case, the computed value of the discriminant differed from an integer by less than 0.001.

The results of the computation are summarized in two tables. In Table 1, a list of the 96 equations is given, showing, in each case, the span, the gap (that is, the minimum distance between two roots), and the largest and smallest roots, all to four decimals, and finally the coefficients and the discriminant, with its factorization into primes. The equations are arranged first according to degree, and then by increasing span. The equations are identified by symbols $2a, 2b, \cdots, 8z$, where the number is the degree, and the letter indicates the rank in the list by increasing span. In Table 2, all of the roots of the equations are given to ten decimals, the equations being identified by their symbols.

Of the 96 equations listed in Table 1, 19 correspond to Kronecker polynomials

$$P_m(x) = \prod_{0 \leq k \leq m/2; (k,m)=1} (x - 2 \cos 2k\pi/m),$$

which have all their roots in $[-2, 2]$. The equations, and the corresponding values of m, are as follows:

No.	$2a$,	$2b$,	$2c$,	$3a$,	$3b$,	$4c$,	$4e$,	$4f$,	$4i$,	$5a$,	$6a$,	$6g$,	$6i$,	$6k$,	$8b$,	$8e$,	$8k$,	$8p$,	$8r$.
$m =$	10,	8,	12,	14,	18,	16,	15,	20,	24,	22,	26,	21,	28,	36,	34,	32,	40,	48,	60.

Notice that if we change the sign of the roots of $P_m(x)$, then we double m if m is odd, halve m if m is twice an odd number, and leave m unchanged if m is a multiple of 4. When $m = 10$, it is also possible to diminish the roots by 1, but this has the same effect as changing the signs. Thus the polynomials $P_m(x)$ not listed in Table 1, but equivalent to entries there, are those with $m = 5, 7, 9, 11, 13, 17, 30, 42$. We should perhaps add that $P_m(x)$ has degree 1 when $m = 1, 2, 3, 4, 6$, and degree 9 or greater in all cases which have not been mentioned.

It is known that the cyclotomic polynomial $Q_m(t) = t^{\phi(m)} + \cdots$, whose roots are the primitive mth roots of 1, has the discriminant

$$(-1)^{\phi(m)/2} m^{\phi(m)} : \prod_{p \mid m} p^{\phi(m)/(p-1)}$$

TABLE 1

List of Algebraic Equations with Span Less than 4

No.	Span Gap	Max. Rt. Min. Rt.	Coefficients Discriminant
$2a$	2.2361 2.2361	1.6180 −0.6180	$1 - 1 - 1$ $5 = \text{prime}$
$2b$	2.8284 2.8284	1.4142 −1.4142	$1 + 0 - 2$ $8 = 2^3$
$2c$	3.4641 3.4641	1.7321 −1.7321	$1 + 0 - 3$ $12 = 2^2 \cdot 3$
$2d$	3.6056 3.6056	2.3028 −1.3028	$1 - 1 - 3$ $13 = \text{prime}$
$3a$	3.0489 1.3569	1.8019 −1.2470	$1 - 1 - 2 + 1$ $49 = 7^2$
$3b$	3.4115 1.1848	1.8794 −1.5321	$1 + 0 - 3 - 1$ $81 = 3^4$
$3c$	3.6513 1.7923	2.1701 −1.4812	$1 - 1 - 3 + 1$ $148 = 2^2 \cdot 37$
$3d$	3.8895 1.1359	2.2143 −1.6751	$1 + 0 - 4 - 2$ $148 = 2^2 \cdot 37$
$3e$	3.9757 1.6067	2.1149 −1.8608	$1 + 0 - 4 - 1$ $229 = \text{prime}$
$4a$	3.3871 0.8986	2.1935 −1.1935	$1 - 2 - 2 + 3 + 1$ $725 = 5^2 \cdot 29$
$4b$	3.4510 0.8784	2.0953 −1.3557	$1 - 1 - 3 + 1 + 1$ $725 = 5^2 \cdot 29$
$4c$	3.6955 1.0824	1.8478 −1.8478	$1 + 0 - 4 + 0 + 2$ $2048 = 2^{11}$
$4d$	3.7032 0.8289	2.1268 −1.5764	$1 - 1 - 4 + 2 + 3$ $1957 = 19 \cdot 103$
$4e$	3.7834 0.4888	1.8271 −1.9563	$1 - 1 - 4 + 4 + 1$ $1125 = 3^2 \cdot 5^3$
$4f$	3.8042 0.7265	1.9021 −1.9021	$1 + 0 - 5 + 0 + 5$ $2000 = 2^4 \cdot 5^3$
$4g$	3.8255 1.0702	2.0615 −1.7640	$1 + 0 - 4 - 1 + 1$ $1957 = 19 \cdot 103$
$4h$	3.8592 0.7474	2.2631 −1.5962	$1 - 2 - 3 + 5 + 1$ $2777 = \text{prime}$
$4i$	3.8637 1.0353	1.9319 −1.9319	$1 + 0 - 4 + 0 + 1$ $2304 = 2^8 \cdot 3^2$
$4j$	3.8708 0.8288	2.3623 −1.5085	$1 - 1 - 4 + 1 + 2$ $2777 = \text{prime}$
$4k$	3.9279 0.5657	2.1439 −1.7840	$1 + 0 - 5 - 1 + 4$ $2777 = \text{prime}$
$4l$	3.9335 0.6892	2.4667 −1.4667	$1 - 2 - 4 + 5 + 5$ $2525 = 5^2 \cdot 101$

TABLE 1—*Continued*

No.	Span / Gap	Max. Rt. / Min. Rt.	Coefficients / Discriminant
4*m*	3.9644 0.8744	2.5231 −1.4413	1 − 1 − 4 + 0 + 1 1957 = 19·103
4*n*	3.9910 1.2758	2.4955 −1.4955	1 − 2 − 3 + 4 + 1 4752 = $2^4 \cdot 3^3 \cdot 11$
5*a*	3.6015 0.6093	1.9190 −1.6825	1 − 1 − 4 + 3 + 3 − 1 14 641 = 11^4
5*b*	3.7396 0.4197	2.0431 −1.6965	1 + 0 − 5 − 1 + 5 + 1 24 217 = 61·397
5*c*	3.7645 0.4931	2.1064 −1.6582	1 − 1 − 5 + 3 + 6 − 1 36 497 = prime
5*d*	3.7760 0.4333	2.2242 −1.5518	1 − 2 − 4 + 7 + 4 − 5 24 217 = 61·397
5*e*	3.7852 0.6208	2.4498 −1.3354	1 − 2 − 3 + 4 + 2 − 1 24 217 = 61·397
5*f*	3.8439 0.6789	2.1388 −1.7050	1 − 2 − 3 + 6 + 0 − 1 24 217 = 61·397
5*g*	3.8743 0.9167	2.3180 −1.5563	1 − 2 − 3 + 5 + 1 − 1 36 497 = prime
5*h*	3.9130 0.4177	1.8866 −2.0264	1 − 1 − 5 + 5 + 4 − 3 36 497 = prime
5*i*	3.9222 0.7285	2.0851 −1.8371	1 − 1 − 5 + 4 + 5 − 3 65 657 = prime
5*j*	3.9469 0.3327	2.4039 −1.5430	1 − 1 − 5 + 2 + 5 − 1 38 569 = prime
5*k*	3.9635 0.4993	2.3283 −1.6353	1 − 2 − 4 + 6 + 4 − 1 81 509 = prime
5*l*	3.9757 0.3711	2.0541 −1.9216	1 + 0 − 6 + 0 + 8 − 1 81 589 = 83·983
5*m*	3.9797 0.6950	2.3928 −1.5869	1 − 2 − 4 + 6 + 4 − 2 126 032 = $2^4 \cdot 7877$
5*n*	3.9821 0.5726	2.2684 −1.7138	1 − 1 − 5 + 3 + 5 − 2 81 509 = prime
5*o*	3.9926 0.5149	2.0385 −1.9541	1 + 0 − 5 + 0 + 4 − 1 38 569 = prime
6*a*	3.7128 0.4449	1.9419 −1.7709	1 − 1 − 5 + 4 + 6 − 3 − 1 371 293 = 13^5
6*b*	3.7400 0.5265	2.3700 −1.3700	1 − 3 − 2 + 9 + 1 − 6 − 1 434 581 = $7^4 \cdot 181$
6*c*	3.7970 0.5006	2.2831 −1.5140	1 − 3 − 2 + 10 − 1 − 7 + 1 485 125 = $5^3 \cdot 3881$
6*d*	3.8077 0.3869	2.1588 −1.6490	1 − 2 − 4 + 7 + 4 − 4 − 1 592 661 = prime

TABLE 1—*Continued*

No.	Span / Gap	Max. Rt. / Min. Rt.	Coefficients / Discriminant
6e	3.8633 0.5392	2.4316 −1.4316	1 − 3 − 2 + 9 + 0 − 5 + 1 810 448 = $2^4 \cdot 37^3$
6f	3.8865 0.3177	2.0702 −1.8163	1 − 1 − 6 + 5 + 9 − 6 − 1 1 202 933 = 79·15 227
6g	3.8888 0.2587	1.9111 −1.9777	1 − 1 − 6 + 6 + 8 − 8 + 1 453 789 = $3^3 \cdot 7^5$
6h	3.8895 0.3331	2.4448 −1.4448	1 − 3 − 3 + 11 + 3 − 9 − 1 1 528 713 = $3^8 \cdot 233$
6i	3.8997 0.3862	1.9499 −1.9499	1 + 0 − 7 + 0 + 14 + 0 − 7 1 075 648 = $2^6 \cdot 7^5$
6j	3.9293 0.4378	2.1233 −1.8060	1 − 2 − 4 + 8 + 2 − 5 + 1 485 125 = $5^3 \cdot 3881$
6k	3.9392 0.6015	1.9696 −1.9696	1 + 0 − 6 + 0 + 9 + 0 − 3 1 259 712 = $2^6 \cdot 3^9$
6l	3.9427 0.3024	2.3736 −1.5691	1 − 2 − 5 + 7 + 8 − 3 − 1 1 241 125 = $5^3 \cdot 9929$
6m	3.9456 0.5345	2.0467 −1.8989	1 − 1 − 5 + 4 + 5 − 2 − 1 592 661 = prime
6n	3.9457 0.3125	2.2552 −1.6906	1 − 2 − 5 + 9 + 6 − 9 + 1 1 416 125 = $5^3 \cdot 11\,329$
6o	3.9507 0.4485	2.3448 −1.6059	1 − 2 − 5 + 8 + 8 − 7 − 4 1 868 969 = 107·17 467
6p	3.9979 0.4211	2.2050 −1.7929	1 − 1 − 6 + 4 + 9 − 3 − 1 3 086 597 = 383·8059
6q	3.9986 0.2324	2.4249 −1.5737	1 − 2 − 5 + 8 + 7 − 7 − 1 2 286 997 = 349·6553
7a	3.8630 0.3407	2.2788 −1.5842	1 − 2 − 5 + 8 + 8 − 7 − 3 + 1 20 134 393 = 71·283 583
7b	3.8646 0.3444	2.3912 −1.4733	1 − 3 − 3 + 12 + 2 − 13 + 0 + 3 25 164 057 = 3·8 388 019
7c	3.8809 0.2639	2.0889 −1.7921	1 − 1 − 7 + 5 + 15 − 6 − 9 + 1 41 153 941 = prime
7d	3.8853 0.3173	2.2412 −1.6441	1 − 2 − 5 + 9 + 7 − 10 − 2 + 1 25 164 057 = 3·8 388 019
7e	3.9022 0.3001	2.1532 −1.7490	1 − 2 − 5 + 9 + 7 − 9 − 3 + 1 28 118 369 = prime
7f	3.9243 0.4317	2.2018 −1.7225	1 − 1 − 6 + 4 + 10 − 4 − 4 + 1 20 134 393 = 71·283 583
7g	3.9251 0.2437	2.5366 −1.3885	1 − 3 − 3 + 11 + 3 − 10 − 1 + 1 32 354 821 = prime
7h	3.9323 0.2532	2.1867 −1.7456	1 − 1 − 7 + 4 + 15 − 2 − 8 − 1 43 242 544 = $2^4 \cdot 101 \cdot 26\,759$
7i	3.9373 0.4349	2.2756 −1.6617	1 − 3 − 3 + 13 + 0 − 14 + 2 + 3 30 653 489 = 1399·21 911

TABLE 1—*Continued*

No.	Span Gap	Max. Rt. Min. Rt.	Coefficients Discriminant
7*j*	3.9600 0.3119	2.0768 −1.8832	1 − 1 − 7 + 5 + 15 − 5 − 10 − 1 39 829 313 = 373·106 781
7*k*	3.9704 0.1028	2.0341 −1.9363	1 + 0 − 8 + 0 + 19 + 0 − 12 − 1 34 554 953 = prime
7*l*	3.9713 0.3383	2.2101 −1.7612	1 − 2 − 6 + 11 + 11 − 17 − 6 + 7 41 455 873 = 37·1 120 429
7*m*	3.9754 0.1724	2.3144 −1.6610	1 − 1 − 7 + 4 + 15 − 3 − 9 − 1 35 269 513 = prime
7*n*	3.9824 0.2364	2.0780 −1.9043	1 − 1 − 7 + 6 + 13 − 9 − 3 + 1 39 610 073 = 89·599·743
7*o*	3.9980 0.3251	2.4775 −1.5206	1 − 3 − 3 + 11 + 3 − 9 − 2 + 1 32 567 681 = 89·365 929
8*a*	3.7977 0.2474	2.3989 −1.3989	1 − 4 − 1 + 17 − 5 − 23 + 6 + 9 − 1 309 593 125 = $5^4 \cdot 19 \cdot 29^2 \cdot 31$
8*b*	3.8309 0.2655	1.9659 −1.8649	1 − 1 − 7 + 6 + 15 − 10 − 10 + 4 + 1 410 338 673 = 17^7
8*c*	3.9122 0.2248	2.4561 −1.4561	1 − 4 − 1 + 17 − 6 − 21 + 8 + 6 − 1 1 459 172 469 = $3 \cdot 19^2 \cdot 103^2 \cdot 127$
8*d*	3.9186 0.2723	2.4288 −1.4898	1 − 3 − 4 + 14 + 6 − 19 − 5 + 6 + 1 1 348 097 653 = $7^2 \cdot 27\ 512\ 197$
8*e*	3.9231 0.2986	1.9616 −1.9616	1 + 0 − 8 + 0 + 20 + 0 − 16 + 0 + 2 2 147 483 648 = 2^{31}
8*f*	3.9294 0.2923	2.0745 −1.8549	1 + 0 − 8 − 1 + 20 + 4 − 16 − 3 + 2 1 359 341 129 = prime
8*g*	3.9337 0.2709	2.3339 −1.5998	1 − 3 − 4 + 15 + 4 − 22 + 0 + 9 − 1 1 391 339 501 = 71·149·131 519
8*h*	3.9380 0.2334	2.3788 −1.5593	1 − 3 − 4 + 15 + 4 − 22 + 0 + 8 − 1 1 299 600 812 = $2^2 \cdot 324\ 900\ 203$
8*i*	3.9415 0.2393	2.2321 −1.7094	1 − 2 − 6 + 11 + 11 − 17 − 6 + 6 + 1 1 348 097 653 = $7^2 \cdot 27\ 512\ 197$
8*j*	3.9465 0.2745	2.2893 −1.6573	1 − 2 − 6 + 10 + 12 − 13 − 8 + 3 + 1 1 299 600 812 = $2^2 \cdot 324\ 900\ 203$
8*k*	3.9508 0.1934	1.9754 −1.9754	1 + 0 − 8 + 0 + 19 + 0 − 12 + 0 + 1 1 024 000 000 = $2^{16} \cdot 5^6$
8*l*	3.9531 0.1320	2.3871 −1.5661	1 − 4 − 2 + 21 − 6 − 33 + 12 + 13 − 1 1 077 044 573 = 11·97 913 143
8*m*	3.9537 0.0856	2.2356 −1.7181	1 − 3 − 5 + 18 + 7 − 33 − 3 + 18 + 1 707 295 133 = 9973·70 921
8*n*	3.9627 0.2508	2.1504 −1.8123	1 − 2 − 6 + 11 + 11 − 16 − 7 + 5 + 1 1 621 897 976 = $2^3 \cdot 239 \cdot 848\ 273$
8*o*	3.9649 0.2369	2.2215 −1.7434	1 − 1 − 7 + 5 + 15 − 7 − 10 + 2 + 1 661 518 125 = $5^4 \cdot 439 \cdot 2411$

TABLE 1—*Continued*

No.	Span / Gap	Max. Rt. / Min. Rt.	Coefficients / Discriminant
8p	3.9658 0.3692	1.9829 −1.9829	1 + 0 − 8 + 0 + 20 + 0 − 16 + 0 + 1 1 358 954 496 = $2^{24}\cdot 3^4$
8q	3.9777 0.2130	2.2921 −1.6857	1 − 3 − 4 + 15 + 4 − 21 − 2 + 8 + 1 1 442 599 461 = $3^2\cdot 29\cdot 2351^2$
8r	3.9781 0.3976	1.9890 −1.9890	1 + 0 − 7 + 0 + 14 + 0 − 8 + 0 + 1 324 000 000 = $2^8\cdot 3^4\cdot 5^6$
8s	3.9815 0.4274	2.1461 −1.8354	1 − 1 − 7 + 5 + 15 − 6 − 10 + 1 + 1 1 424 875 717 = 15 493·91 969
8t	3.9827 0.2028	2.2589 −1.7238	1 − 2 − 7 + 12 + 17 − 21 − 16 + 10 + 5 2 121 175 625 = $5^4\cdot$3 393 881
8u	3.9841 0.2001	2.2753 −1.7088	1 − 2 − 7 + 11 + 18 − 15 − 18 + 0 + 1 1 077 044 573 = 11·97 913 143
8v	3.9858 0.2397	2.2853 −1.7005	1 − 2 − 7 + 12 + 17 − 21 − 15 + 9 + 1 2 256 866 749 = prime
8w	3.9865 0.2625	2.2306 −1.7559	1 − 1 − 8 + 5 + 21 − 6 − 18 + 2 + 3 1 798 549 237 = prime
8x	3.9893 0.2510	2.3816 −1.6078	1 − 4 − 1 + 18 − 9 − 21 + 13 + 3 − 1 1 994 682 269 = 1877·1 062 697
8y	3.9955 0.2647	2.2974 −1.6981	1 − 1 − 7 + 4 + 15 − 3 − 9 + 0 + 1 483 345 053 = prime
8z	3.9973 0.1868	2.3517 −1.6456	1 − 2 − 7 + 11 + 18 − 16 − 18 + 3 + 3 2 495 071 701 = $3^3\cdot$1303·70 921

for $m > 2$, where p runs through the prime divisors of m. A proof may be found in E. Lehmer [2]. Since $Q_m(t) = t^{\phi(m)/2}P_m(t + 1/t)$ for $m > 2$, this result can be used to find the discriminant of $P_m(x)$. Indeed, it is easily seen that the discriminant of $P_m(x)$ for $m > 2$ is equal to

$$\left(m^{\phi(m)} : c\prod_{p\mid m} p^{\phi(m)/(p-1)}\right)^{1/2}$$

where

$$c = \begin{cases} p & \text{if} \quad m = p^l,\ 2p^l \qquad (p > 2), \\ 4 & \text{if} \quad m = 2^l, \\ 1 & \text{otherwise.} \end{cases}$$

Here p denotes a prime, and l a positive integer. The computed values agree with those found from this formula. The formula shows that the discriminants of the Kronecker polynomials are very round numbers. A glance at Table 1 shows that most of the other discriminants are not so round; indeed, many of them are prime.

Another observation is that many of the discriminants occur repeatedly. For example, there are four different quintics with discriminant 24217.

An interesting fact that does not show in Table 1, but was apparent in the preliminary output, is that there is a strong correlation between the smallness of the gap between the nearest roots of a polynomial, and the reducibility of the poly-

TABLE 2

Roots of Algebraic Equations with Span less than 4

2*a* 1.61803 39887 −0.61803 39887	2*b* 1.41421 35624 −1.41421 35624	2*c* 1.73205 08076 −1.73205 08076	2*d* 2.30277 56377 −1.30277 56377
3*a* 1.80193 77358 0.44504 18679 −1.24697 96037	3*b* 1.87938 52416 −0.34729 63553 −1.53208 88862	3*c* 2.17008 64866 0.31110 78175 −1.48119 43041	3*d* 2.21431 97434 −0.53918 88728 −1.67513 08706
3*e* 2.11490 75415 −0.25410 16884 −1.86080 58531	4*a* 2.19352 70853 1.29496 28993 −0.29496 28993 −1.19352 70853	4*b* 2.09529 39852 0.73764 03052 −0.47725 99965 −1.35567 42940	4*c* 1.84775 90650 0.76536 68647 −0.76536 68647 −1.84775 90650
4*d* 2.12675 70596 1.19712 62976 −0.74746 84539 −1.57641 49033	4*e* 1.82709 09153 1.33826 12127 −0.20905 69265 −1.95629 52015	4*f* 1.90211 30326 1.17557 05046 −1.17557 05046 −1.90211 30326	4*g* 2.06149 88507 0.39633 85310 −0.69382 24565 −1.76401 49252
4*h* 2.26307 74103 1.51572 15893 −0.18264 43236 −1.59615 46760	4*i* 1.93185 16526 0.51763 80902 −0.51763 80902 −1.93185 16526	4*j* 2.36233 98329 0.82578 45519 −0.67964 31856 −1.50848 11992	4*k* 2.14386 44258 0.85844 19548 −1.21830 96530 −1.78399 67275
4*l* 2.46673 18040 1.77748 42509 −0.77748 42509 −1.46673 18040	4*m* 2.52309 55906 0.48508 39474 −0.56688 86276 −1.44129 09104	4*n* 2.49550 76566 1.21968 68711 −0.21968 68711 −1.49550 76566	5*a* 1.91898 59472 1.30972 14679 0.28462 96765 −0.83083 00260 −1.68250 70657
5*b* 2.04314 09156 1.13012 71160 −0.19993 34460 −1.27683 96986 −1.69649 48871	5*c* 2.10636 97048 1.55929 59532 0.15759 67325 −1.16509 43780 −1.65816 80124	5*d* 2.22418 46483 1.67128 35260 0.77490 94377 −1.11854 26729 −1.55183 49391	5*e* 2.44982 94641 1.26073 41678 0.33932 84354 −0.71453 30635 −1.33535 90038
5*f* 2.13882 90696 1.45989 17376 0.48980 28469 −0.38348 42661 −1.70503 93880	5*g* 2.31801 35147 1.33418 51529 0.41037 53841 −0.50628 72600 −1.55628 67917	5*h* 1.88660 96452 1.46887 60099 0.58466 39955 −0.91373 17013 −2.02641 79492	5*i* 2.08508 11918 1.32892 33446 0.53163 32638 −1.10855 64842 −1.83708 13160
5*j* 2.40387 48019 1.15694 20520 0.19253 13863 −1.21033 19657 −1.54301 62745	5*k* 2.32828 68228 1.82900 88824 0.19912 64589 −0.72115 91569 −1.63526 30071	5*l* 2.05410 71518 1.29150 07115 0.12651 46910 −1.55052 65930 −1.92159 59613	5*m* 2.39275 62472 1.69774 65488 0.35947 02421 −0.86306 70539 −1.58690 59843

TABLE 2—*Continued*

5*n*	5*o*	6*a*	6*b*
2.26835 33165	2.03849 52910	1.94188 36349	2.37002 12843
1.21567 55100	0.79073 43035	1.49702 14963	1.84348 73025
0.37086 49091	0.27583 41933	0.70920 97741	1.16937 45321
−1.14112 85272	−1.15098 41733	−0.24107 33605	−0.16937 45321
−1.71376 52084	−1.95407 96146	−1.13612 94935	−0.84348 73025
		−1.77091 20513	−1.37002 12843
6*c*	6*d*	6*e*	6*f*
2.28306 70725	2.15875 16408	2.43162 99604	2.07015 94250
1.75190 56915	1.77183 58725	1.89244 10876	1.58408 97733
1.25135 09117	0.81582 30347	0.78268 97829	0.80055 70644
0.14401 29992	−0.22199 32171	0.21731 02171	−0.13994 65390
−0.91638 39954	−0.87542 74923	−0.89244 10876	−1.49856 60766
−1.51395 26794	−1.64898 98386	−1.43162 99604	−1.81629 36470
6*g*	6*h*	6*i*	6*j*
1.91114 56116	2.44475 93389	1.94985 58244	2.12327 32553
1.65247 75486	2.11161 29670	1.56366 29649	1.68547 18076
0.73068 20487	1.10878 13716	0.86776 74782	0.68273 21397
0.14946 01872	−0.10878 13716	−0.86776 74782	0.24385 63775
−1.46610 37437	−1.11161 29670	−1.56366 29649	−0.92932 81851
−1.97766 16525	−1.44475 93389	−1.94985 58244	−1.80600 53949
6*k*	6*l*	6*m*	6*n*
1.96961 55060	2.37360 03535	2.04670 58732	2.25517 13921
1.28557 52194	2.07124 50231	1.38700 01779	1.85635 10103
0.68404 02867	0.49785 73152	0.65570 47692	0.83422 15005
−0.68404 02867	−0.22710 87607	−0.32801 78979	0.12290 56033
−1.28557 52194	−1.14649 15929	−0.86250 94037	−1.37807 69953
−1.96961 55060	−1.56910 23383	−1.89888 35188	−1.69057 25109
6*o*	6*p*	6*q*	7*a*
2.34475 18515	2.20497 70881	2.42486 15948	2.27880 70929
1.76565 60212	1.64986 29905	1.75560 15024	1.93806 50941
1.11792 22833	0.52342 27917	0.86345 85959	0.87572 24244
−0.46496 59997	−0.21351 89504	−0.12887 98831	0.24207 93268
−1.15744 64577	−1.37181 98850	−1.34132 18022	−0.57219 13441
−1.60591 76986	−1.79292 40351	−1.57372 00078	−1.17827 35857
			−1.58420 90085
7*b*	7*c*	7*d*	7*e*
2.39123 75594	2.08888 35693	2.24120 40051	2.15315 39967
1.91325 34641	1.82496 98929	1.70295 89324	1.85310 01391
1.23962 22465	1.09719 47423	1.17760 79154	1.14835 13343
0.59378 69656	0.10569 07208	0.25271 46462	0.21950 92371
−0.53556 36555	−0.86443 06713	−0.40360 44121	−0.50960 73144
−1.12898 87698	−1.46024 96100	−1.32677 95062	−1.11551 11099
−1.47334 78104	−1.79205 86440	−1.64410 15808	−1.74899 62829

Table 2—*Continued*

7*f*	7*g*	7*h*	7*i*
2.20183 39730	2.53663 40001	2.18667 40539	2.27555 45388
1.51066 11464	1.95021 60042	1.93346 91500	1.84067 40073
0.80437 50613	1.13664 55841	0.93191 78785	1.36914 49724
0.22934 60487	0.29238 53908	−0.13376 41384	0.68007 32224
−0.73300 93107	−0.38268 33351	−0.78108 16535	−0.43188 29664
−1.29073 24447	−1.14473 53768	−1.39163 10759	−1.07182 01568
−1.72247 44739	−1.38846 22674	−1.74558 42146	−1.66174 36178

7*j*	7*k*	7*l*	7*m*
2.07680 91667	2.03413 68479	2.21007 56822	2.31441 11031
1.76493 74245	1.65062 67108	1.87175 87733	1.68056 07022
1.27273 81755	1.08595 83522	1.23180 73173	1.08337 53571
−0.10757 69753	−0.08427 83080	0.66279 18687	−0.11845 28845
−0.79832 42651	−0.91659 32836	−0.88404 60243	−0.81025 58461
−1.32536 65502	−1.83353 73834	−1.33116 55726	−1.48862 73940
−1.88321 69762	−1.93631 29360	−1.76122 20446	−1.66101 10378

7*n*	7*o*	8*a*	8*b*
2.07804 57594	2.47747 11727	2.39886 13151	1.96594 61994
1.81563 43448	2.05363 38895	2.15144 17932	1.70043 42715
0.86539 15031	1.12041 46295	1.72978 03441	1.20526 92728
0.23086 84170	0.25444 72868	0.89332 68549	0.54732 59801
−0.41764 84063	−0.53013 58647	0.10667 31451	−0.18453 67189
−1.66796 97214	−0.85525 86034	−0.72978 03441	−0.89147 67116
−1.90432 18966	−1.52057 25103	−1.15144 17932	−1.47801 78344
		−1.39886 13151	−1.86494 44588

8*c*	8*d*	8*e*	8*f*
2.45612 24152	2.42876 01763	1.96157 05608	2.07451 94192
2.23131 98589	2.05053 08962	1.66293 92246	1.69234 74357
1.52609 63417	1.49093 45592	1.11114 04660	1.07758 22341
0.85105 97391	0.63384 55510	0.39018 06440	0.29176 06409
0.14894 02609	−0.15872 53449	−0.39018 06440	−0.52266 88171
−0.52609 63417	−0.73796 61606	−1.11114 04660	−1.19608 43121
−1.23131 98589	−1.21756 33176	−1.66293 92246	−1.56258 38332
−1.45612 24152	−1.48981 63597	−1.96157 05608	−1.85487 27675

8*g*	8*h*	8*i*	8*j*
2.33390 51943	2.37877 05280	2.23208 53685	2.28928 02351
2.06295 99589	1.92985 82731	1.81608 09200	1.96017 22617
1.36617 23543	1.53431 37336	1.26804 86993	1.23336 56728
0.79662 21523	0.68111 13800	0.71743 97852	0.47533 09359
0.11468 99392	0.13096 00598	−0.15414 60600	−0.24726 15874
−0.84795 88071	−0.76991 00007	−0.70002 37500	−0.67083 22418
−1.22662 28613	−1.32584 28197	−1.47011 58178	−1.38279 04401
−1.59976 79307	−1.55926 11542	−1.70936 91452	−1.65726 48363

8*k*	8*l*	8*m*	8*n*
1.97537 66812	2.38706 40662	2.23559 76095	2.15041 14940
1.78201 30484	2.23910 60007	2.15002 67393	1.89964 35301
0.90798 09995	1.80781 37653	1.58578 29306	1.36653 64300
0.31286 89301	1.07846 71566	1.09198 22828	0.59805 20552
−0.31286 89301	0.07300 11789	−0.05535 89433	−0.17581 47903
−0.90798 09995	−0.58531 69585	−0.92643 79819	−0.71891 58705
−1.78201 30484	−1.43404 98300	−1.36345 90650	−1.30764 92408
−1.97537 66812	−1.56608 53792	−1.71813 35719	−1.81226 36076

TABLE 2—*Continued*

$8o$	$8p$	$8q$	$8r$
2.22153 24876	1.98288 97227	2.29207 90117	1.98904 37907
1.55561 23225	1.58670 66806	2.07909 08161	1.48628 96510
1.16052 22500	1.21752 28580	1.43437 77844	0.81347 32862
0.42908 59241	0.26105 23844	0.85052 13404	0.41582 33816
−0.25745 81743	−0.26105 23844	−0.12637 03068	−0.41582 33816
−0.85931 74559	−1.21752 28580	−0.71552 88820	−0.81347 32862
−1.50656 25745	−1.58670 66806	−1.12851 51069	−1.48628 96510
−1.74341 47794	−1.98288 97227	−1.68565 46568	−1.98904 37907
$8s$	$8t$	$8u$	$8v$
2.14606 73652	2.25891 73222	2.27526 75277	2.28531 36141
1.71870 71300	2.05610 77585	2.07518 51952	1.92850 54268
1.11900 47370	1.32533 98244	1.66866 42589	1.59579 07216
0.36899 54337	0.87190 42270	0.22173 55606	0.53854 47058
−0.31650 62081	−0.40705 09516	−0.28125 07082	−0.09757 05446
−0.81640 94247	−0.89051 64126	−0.85062 92510	−1.08935 57382
−1.38446 90891	−1.49093 11702	−1.40016 57332	−1.46075 51746
−1.83538 99441	−1.72377 05976	−1.70880 68502	−1.70047 30109
$8w$	$8x$	$8y$	$8z$
2.23056 99297	2.38158 76097	2.29737 50522	2.35172 16719
1.96804 19244	2.13059 68863	1.75862 05855	2.04492 34889
1.08244 63149	1.64839 12780	0.90472 96661	1.53038 81076
0.50902 52332	0.84150 09960	0.35035 54513	0.45824 16131
−0.41744 54453	0.21026 94526	−0.42535 04548	−0.44139 25348
−1.16793 35980	−0.32962 78318	−0.75418 45108	−0.83950 49209
−1.44876 59545	−1.27495 72375	−1.43342 86594	−1.45879 57700
−1.75593 84043	−1.60776 11534	−1.69811 71301	−1.64558 16558

nomial. To take a specific case, we see from Table 1 that a sextic equation

$$x^6 + \cdots = 0$$

with integer coefficients and real roots, for which the span is less than 4 and the gap is less than 0.23, is reducible. Indeed, there are just 17 essentially different irreducible sextics with span less than 4, and the gaps vary from 0.6015 down to 0.2324. On the other hand, there are many reducible sextics with span less than 4 having smaller gaps. Considering just the sextics which are the product of two different irreducible cubics, then our preliminary output showed that there are 19 inequivalent ones. (These could also be found by combining the five inequivalent cubics, and cubics equivalent to them, in all possible ways.) These reducible sextics fall into two groups: 7 with gaps from 0.5550 down to 0.3111, which look no different from the irreducible sextics in this respect, and 12 with gaps from 0.1430 down to 0.0186, whose gaps are too small for irreducible sextics. It would be interesting to find a general theorem relating the smallness of the gap between the nearest roots of a polynomial to its reducibility.

University of California
Berkeley, California

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